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The Solution of Some Turning Point Problems

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THE SOLUTION OF SOME TURNING POINT PROBLEMS

by

Herbert Kurss

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Table of Contents

Abstract i

I. Introduction 1

II. Perturbation of a Diagonal System 11

III. The Second Order Equation 22

A. Introduction 22

B. Reduction to the Canonical System 25

C. Diagonalization of the Canonical System . . 27

IV. A Fourth Order Equation 34

V. Conclusions 42

Appendix A: The Hypergeometric Equation . . . 43

Appendix B: Stokes Multipliers 47

Bibliography 50

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The Solution of Some Turning Point Problems

(Abstract)

This report solves some turning point problems for certain systems of the form

$$\frac{d}{dx} u = [\rho A(x,\rho) + B(x,\rho)]u$$

where ρ is a large parameter and $A(x,\rho)$ and $B(x,\rho)$ are square matrices analytic in x and ρ at $x = 0$ and $\rho = \infty$.

The basic approach is to obtain connection formula for u in terms of solutions to the comparison equation

$$\frac{d}{dx} G = \rho A(x,\rho)G$$

For this purpose, define

$$\frac{d}{dx} E = \rho J E$$

and

$$V = GE^{-1}$$

and let

$$u = Vw$$

Then

$$\frac{d}{dx} w = [\rho J + V^{-1}BV]w$$

Assume that J is diagonal and in some sector (of the complex x-plane) with vertex at $x = 0$ the elements of J and $xV^{-1}BV$ are Hölder continuous. Connection formula for w in the sector are then obtained by an elaboration of a scheme due to K. O. Friedrichs.

Two natural comparison systems are

$$\frac{d}{dx} G = \rho \begin{Bmatrix} \delta_{i+1,j} x^{a_i}, & i \neq n \\ \delta_{ij} x^{a_n}, & i = n \end{Bmatrix} G$$

and

$$\frac{d}{dx} G = \rho \begin{Bmatrix} \delta_{i+1,j} x^{a_i}, & i \neq n \\ \delta_{2j} x^{a_n}, & i = n \end{Bmatrix} G$$

where each a_i is a non-negative integer. In each case the elements of G can be written explicitly in terms of generalized hypergeometric functions. These comparison systems are applied, respectively, to a second order equation and a fourth order equation of the type which arises in the theory of hydrodynamic stability. In each case the integrability condition on $V^{-1}BV$ is verified. A typical result is to obtain connection formula for the second order equation

$$\frac{d^2}{dx^2} u = [\rho^2 a(x) + \rho b(x) + c(x,\rho)]u$$

where the order of the zero of $a(x)$ and $b(x)$ at $x = 0$ is, respectively, α and β, in terms of connection formula for

$$\frac{d^2}{dx^2} g = \rho^2 x^{\alpha} g$$

whenever

$$\beta > \frac{\alpha}{2} - 1 .$$

I. Introduction

The solution of turning point problems via the connection approach (which will be described in a moment) proceeds in two parts. (See, for example*, Wasow [2,Chap.6]).

In the first part one obtains connection formula for solutions of a special system by a contour integral approach (Watson [1,Chap.7]), or by a Laplace transform approach (Wasow [1], Linstone [1], Lin [1]) or by a Taylor series approach (Scheffé [1], Turrittin [1], Langer [2]).

In the second part first the special system is converted via a 1-1 co-ordinate transformation so as to bear a "sufficiently close" resemblance to the original system. The system which results from this transformation is called a comparison system. Finally, with the aid of a comparison technique, connection formula for the original system are obtained in terms of connection formula for the comparison system.

Needless to say, the finer the comparison technique the cruder can be the comparison system. In this direction, an important improvement in the standard comparison technique was indicated by Friedrichs [1,§22] where, for simplicity, he considered a system of two first order equations. An elaboration and exploitation of this improvement forms the main theme of this report. However, before we discuss the improved comparison technique let us first briefly review some of the vital facts concerning turning point problems.

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* This refers to Chapter 6 in the second publication listed under Wasow in the Bibliography at the end of this report.

The basic system to be considered is

$$\frac{d}{dx}u = [\rho A(x,\rho) + B(x,\rho)]u \tag{1.1}$$

where

u = a column vector of dimension n

$|\rho|$ = a large parameter

x = the complex independent variable ranging over a Riemann surface $\mathcal{R}$.

$A(x,\rho), B(x,\rho)$ = square $n \times n$ matrices analytic in x and ρ at $x = 0$ and $\rho = \infty$.

The basic question to be answered is, "How can one determine the asymptotic behavior ($|\rho| \to \infty$) of a typical solution of (1.1) for all x in $\mathcal{R}$?" Complications arise because of the occurrence of so-called turning points (sometimes called transition points) and in order to grapple with the heart of the problem region $\mathcal{R}$ will be taken to have a turning point at $x = 0$ and no turning point for $x \neq 0$. What this implies for (1.1) can best be described in terms of the Jordan form of the leading matrix $A(x,\infty)$. This Jordan form will be assumed to be diagonal for x in $\mathcal{R}$ but $x \neq 0$.

Let $\lambda_i(x)$, $i = 1,2,\ldots,n$ denote the eigenvalues of $A(x,\infty)$ for $x \neq 0$ and define

$$J(x) = \{\delta_{ij}\lambda_i(x)\} \tag{1.2}$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Then (by the above assumption) for $x \neq 0$, $J(x)$ is the Jordan

form of $A(x,\infty)$. Let $\bar{i}$ denote the set of integers, s, for which $\lambda_s(x) = \lambda_i(x)$, i.e.,

$$\bar{i} = \bar{s} \iff \lambda_i(x) = \lambda_s(x) \quad \text{for all } x \text{ in } \mathscr{R} \tag{1.3}$$

It turns out that no turning point for $x \neq 0$ requires

$$\prod_{\bar{i} \neq \bar{j}} (\lambda_i(x) - \lambda_j(x)) \neq 0 \quad \text{for } x \neq 0 \tag{1.4a}$$

i.e. for $x \neq 0$ eigenvalues of $A(x,\infty)$ which are distinct for some $x \neq 0$ remain distinct for all $x \neq 0$, and a turning point at $x = 0$ requires

$$\prod_{\bar{i} \neq \bar{j}} (\lambda_i(x) - \lambda_j(x)) = 0 \quad \text{for } x = 0 \tag{1.4b}$$

i.e., at least two eigenvalues of $A(x,\infty)$ which are distinct for $x \neq 0$ coalesce at $x = 0$.

It is well known that for $x \neq 0$ a set of n independent formal solutions can be exhibited for (1.1) restricted by (1.2) and (1.4). More precisely, corresponding to the eigenvalue $\lambda_k(x)$ of multiplicity m_k there are m_k independent formal solutions each of the form

$$u(x,\rho) = [\exp \rho h_k(x)] \sum_{j=0}^{\infty} \rho^{-j} v_j(x) \tag{1.5}$$

where

$$h_k(x) = \int_0^x \lambda_k(t)dt \tag{1.6}$$

One can readily verify that the condition that (1.5) be a formal solution to (1.1) allows one to determine the $v_j(x)$ successively. (Birkhoff [1], Schlesinger [1], Noaillon [1],

Tamarkin [1]). Alternately, one can verify that the condition that (1.1) be asymptotically transformed into the diagonal equation

$$\frac{d}{dx} w = [\rho\lambda_k(x) + \ldots]w$$

by

$$u = \left[\sum_{j=0}^{\infty} \rho^{-j} P_j(x) \right] w$$

allows one to determine the $P_j(x)$ successively. (Hukuhara [1], Turrittin [2]). In any event, the recursion formula for the $v_j(x)$ involves the pseudo-inverse of $J(x) - \lambda_k(x)$, namely

$$\begin{Bmatrix} [\lambda_i(x) - \lambda_k(x)]^{-1}\delta_{ij} , & i \notin \overline{k} \\ 0 , & \text{otherwise} \end{Bmatrix}$$

This pseudo-inverse and hence, as it turns out, each $v_j(x)$ (in general) is singular at $x = 0$.

In order to remedy this situation it is natural to attempt to replace (1.5) by a more general formal solution of the form

$$u(x,\rho) = [\exp \rho h_k(x)] \sum_{j=0}^{\infty} v_j(x,\rho) \tag{1.7}$$

The fact that this series is to be an asymptotic series implies that in an appropriate norm, $\| \ \|$,

$$\| \rho^{j\gamma} v_j(x,\rho) \|$$

is bounded for some $\gamma > 0$. Of course, since $x = 0$ is a turning point (1.7) can represent the same solution of (1.1) in at most a sectoral neighborhood of $x = 0$. However, the point of

introducing (1.7) is to obtain a uniform expansion in the sense that the sector will include $x = 0$. (Cherry [1]).

Actually, the series in (1.7) will be constructed to be convergent so that u in (1.7) will be an actual solution to (1.1). The evaluation of the series in (1.7) at $x = 0$, i.e.,

$$u(0,\rho) = \sum_{j=0}^{\infty} v_j(0,\rho) \tag{1.8}$$

will then be an asymptotic expansion of $u(0,\rho)$ and this constitutes a so-called "connection formula". The step from the connection to the turning point to the "continuation around the turning point" proceeds via Stokes Multipliers as described in Appendix B.

In the basic approach for obtaining an expansion of the form in (1.7) or, indeed, of the form in (1.5) one attempts a perturbation (or comparison) scheme which goes back to Liouville [1] and Steckloff [1]. This scheme regards the differential equation (1.1) as being a perturbation of a comparison equation. The differential equation is then converted into an integral equation whose inhomogeneous term is a solution of the comparison equation. More precisely, one first defines a matrix G as a solution to a comparison equation

$$\frac{d}{dx} G = \rho A(x,\rho) G \tag{1.9}$$

i.e., each column of G satisfies the system

$$\frac{d}{dx} g = \rho A(x,\rho) g$$

Then it is easily verified that any solution of the integral equation

$$(1.10)\qquad u(x,\rho) = G(x,\rho)c + G(x,\rho)\int^{x} G^{-1}(t,\rho)B(t,\rho)u(t,\rho)dt$$

where c is a constant vector will also satisfy equation (1.1). What remains to be done is to arrange matters so that in an appropriate norm the iterant solution of the integral equation (1.10) converges according to a negative power of ρ. For this purpose define

$$(1.11)\qquad \frac{d}{dx} E = \rho J E$$

$$(1.12)\qquad V = GE^{-1}$$

and let

$$(1.13)\qquad u = Vw \exp \rho h_k(x)$$

Then it is readily verified that

$$(1.14)\qquad \frac{d}{dx} w = [\rho(J - \lambda_k(x)) + V^{-1}BV]w$$

(Incidentally, in the derivation of equation (1.14) from equations (1.1), (1.11), (1.12), and (1.13) the diagonal nature of J is irrelevant.)

The columns of E must be asymptotically of the same order of magnitude at the connection point $x = 0$ and so one takes

$$(1.15)\qquad E(x,\rho) = \exp \rho \int_{o}^{x} J(t)dt$$

i.e.,

$$E(x,\rho) = \left\{\delta_{ij} \exp \rho h_i(x)\right\}$$

as the appropriate solution of (1.11).

In the standard comparison technique equation (1.14) is compared with

$$\frac{d}{dx} H = \rho(J - \lambda_k(x))H \tag{1.16}$$

to obtain w as the solution of the integral equation

$$w_s = \delta_{ks} + \int_{x_s}^{x} \exp \rho(h_{sk}(x) - h_{sk}(t)) \sum_{j=1}^{n} (V^{-1}BV)_{sj} w_j \, dt \tag{1.17}$$

where

$$h_{sk}(x) = h_s(x) - h_k(x) \tag{1.18}$$

$h_k(x)$ is defined in (1.6)

$\delta_{ks}(x)$ is defined in (1.2)

and the path from x_s to x is one along which $Re\ \rho h_{sk}(x)$ decreases.

The perturbation B is assumed to be so small asymptotically that in an appropriate sector $\Delta(k)$ of the complex x-plane

$$\rho^{\delta} \int_{x_s}^{x} |(V^{-1}BV)_{sj} dt| < \text{const} \tag{1.19}$$

for some $\delta > 0$. The integral operator in (1.17) is then asymptotically contracting in (maximum) norm since the exponential terms in (1.17) are uniformly bounded for all x in $\Delta(k)$ and all t on the paths. (The (maximum) norm of a scalar is a function of ρ and is the maximum with respect to x of its absolute value. The norm of a matrix or a vector is the maximum of the norms of its components.) The iterant solution of (1.17) is then majorized in this (maximum norm by the geometric series

$$\text{const} \sum_{j=0}^{\infty} \rho^{-\delta} .$$

Suppose inequality (1.19) is valid only for $\delta = 0$ and assume, provisionally, that

$$(1.19) \qquad (V^{-1}BV)_{sj} = 0 \quad \text{when } s \in \bar{k} \text{ and } j \in \bar{k}$$

Then the standard comparison technique breaks down but the improved comparison technique comes through by the following appraisals.

A convergence factor is obtained for those terms of (1.17) which contain exponentials in the integrand by an estimate of the type which is customary for Laplace integrals. This estimate is, for $s \notin \bar{k}$,

$$(1.20) \qquad \left| \int_{x_s}^{x} \exp \rho(h_{sk}(x) - h_{sk}(t))(V^{-1}BV)_{sj} dt \right| < C_h |\rho|^{-2\sigma}, \ \sigma > 0$$

and it is to be valid uniformly in $\mathcal{L}(k)$. A sufficient condition for this estimate is that $h_k(x)$ have a Hölder-continuous derivative (which is certainly true if $A(x,\infty)$ is analytic) and $(V^{-1}BV)_{sj}$ be Hölder-integrable, i.e., $x(V^{-1}BV)_{sj}$ be Hölder continuous. Then the original procedure employed by Friedrichs [1,§22] is to introduce a modified norm of the form

$$(1.21) \qquad \|w\|_{\text{mod}} = \sum_{s}^{s \in \bar{k}} \|w_s\| + \rho^{\gamma} \sum_{s}^{s \notin \bar{k}} \|w_s\|$$

where $\| \ \|$ is the above described maximum norm and γ is an appropriate positive constant. It is then a simple matter to show that the integral operator in (1.17) is contracting in this modified norm. A variant of this final step, suggested by L. Nirenberg, is to rewrite the integral operator in (1.17) for

those w_s with $s \in \bar{k}$ by replacing each w_j which occurs in the integrand by its implicit value as given by (1.17). The integral operator in the modified integral equation is then contracting in the original norm.

Let us now return to (1.19). Define projection operator Q as

$$Q = \begin{Bmatrix} \delta_{ij}, & i \in \bar{k} \\ 0 & \text{otherwise} \end{Bmatrix} \tag{1.22}$$

and, for later purposes, the adjoint projection

$$\bar{Q} = \begin{Bmatrix} \delta_{ij}, & i \notin \bar{k} \\ 0 & \text{otherwise} \end{Bmatrix} \tag{1.23}$$

Condition (1.19) can now be rewritten as

$$QV^{-1}BVQ = 0 . \tag{1.24}$$

When (1.24) is not valid instead of comparing (1.14) with (1.16) we rather compare (1.14) with

$$\frac{d}{dx} K = [\rho(J(x) - \lambda_k(x)) + QV^{-1}BVQ]K \tag{1.25}$$

and it is easily seen that the kernel of the integral operator which arises from this comparison will satisfy the decoupling condition implied by (1.24) The improved comparison procedure described above is detailed in Chapter II, and this procedure is applied to special cases in Chapters III and IV.

An implicit assumption in the preceding discussion is that the connection formulas for the comparison equation are either already known or at least are more elementary than the connection

formulas for the original equation. In this respect a natural candidate for the role of comparison equation is

$$\frac{d}{dx} G = \rho \begin{Bmatrix} \delta_{i+1,j}\, x^{\alpha_i}, & i \neq n \\ \delta_{1j}\, x^{\alpha_n}, & i = n \end{Bmatrix} G \tag{1.26}$$

where each α_i is a non-negative integer. By a simple co-ordinate transformation (1.26) can be rewritten as

$$\frac{d}{dt} \hat{G} = \begin{Bmatrix} \delta_{i+1,j}\, t^{\alpha_i}, & i \neq n \\ \delta_{1j}\, t^{\alpha_n}, & i = n \end{Bmatrix} \hat{G} \tag{1.27}$$

The elements of $\hat{G}$ can be written explicitly as (generalized) hypergeometric functions. (See appendix A) Connection formula for (1.27) have been determined by the Taylor series approach only for $\alpha_i = 0$, $i \neq n$ (Scheffé [1], Turrittin [1]) although the same approach would probably work in the general case of (1.27). In terms of G from (1.26) a large class of systems can be reduced to a form amenable to the improved comparison technique. The case n = 2 is relatively simple but rather important and it is discussed in detail in Chapter III. A typical result in Chapter III is to justify the conjecture that connection formula for the scalar equation

$$\frac{d^2u}{dx^2} = [\rho^2 a(x) + \rho b(x) + c(x,\rho)]u \tag{1.28}$$

where as $x \to 0$

$a(x) \to \text{const } x^{\alpha}$, $\alpha > 0$

$b(x) \to \text{const } x^{\beta}$, $\beta \geq 0$

can be obtained in terms of connection formula for

$$\frac{d^2g}{dx^2} = \rho^2 x^{\alpha} g \tag{1.29}$$

whenever

$$\beta > \frac{\alpha}{2} - 1 \tag{1.30}$$

Equation (1.26) can be described as a chain equation of type zero. Another natural comparison equation is

$$\frac{d}{dx} C = \rho \begin{Bmatrix} \delta_{i+1,j}\, x^{\alpha_i}, & i \neq n \\ \delta_{2j}\, x^{\alpha_n}, & i = n \end{Bmatrix} G \tag{1.31}$$

which can be described as a chain equation of type one. For such an equation the first column of G is treated differently from the other columns. This is illustrated in Chapter IV where there is discussed a fourth order equation of the type which arises in the theory of hydrodynamic stability.

II. Perturbation of a Diagonal Equation

It is a surprising fact that a turning point for a differential equation can be created by a small perturbation of the coefficients. This fact was demonstrated by Friedrichs [1,§20] where, for simplicity, the unperturbed equation was

$$\frac{d}{dx} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \rho \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.1}$$

and, for generality, the perturbed equation was taken as

$$(2.2)\qquad \frac{d}{dx}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left[\rho\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Friedrichs [1,§22] described a method for determining inverse Stokes multipliers for equation (2.2) and an elaboration of this method forms the subject matter of this chapter. The discussion here is patterned closely after the discussion in Friedrichs [1,§21,22].

In analogy with equation (2.1) the diagonal equation to be perturbed is

$$(2.3)\qquad \frac{d}{dx}\,u(x,\rho) = \rho J(x)u(x,\rho)$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$|\rho|$ = a large parameter

and, as in (1.2), $J(x)$ is a diagonal matrix with diagonal entries $\lambda_i(x)$, $i = 1,2,\ldots,n$.

Define

$$(2.4)\qquad h_i(x) = \int_0^x \lambda_i(t)dt$$

The columns of the matrix

$$(2.5)\qquad G(x,\rho) = \left\{\delta_{ij} \exp \rho h_i(x)\right\}$$

form a set of n independent solutions to (2.3) and each of these solutions maintains the same exponential factor in a full

neighborhood of every point in the complex x-plane. Accordingly the unperturbed equation (2.3) does not have a turning point.

In general, the perturbed equation

$$\frac{d}{dx}\, u(x,\rho) = [\rho J(x) + P(x,\rho)]u(x,\rho) \tag{2.6}$$

will have turning points even though $P(x,\rho)$ is "bounded" in its dependence upon ρ. This fact will not be demonstrated here. Instead, a turning point shall be presumed to occur at $x = 0$ and a procedure for obtaining a typical connection formula shall be described.

Let the complex independent variable x range over a neighborhood $\mathcal{N}$ surrounding $x = 0$.

The letter k shall be reserved to denote a fixed integer from among the integers from 1 to n inclusive. Let $\bar{k}$ denote, as in (1.3), the class of integers s for which $\lambda_s(x) \equiv \lambda_k(x)$ identically and define the projection operators Q and $\bar{Q}$ as in (1.22) and (1.23).

Let us formulate and demonstrate the key result as a theorem under convenient hypotheses and subsequently discuss the reasonableness of these hypotheses. As a preliminary define

$$h_{sk}(x) = h_s(x) - h_k(x) \tag{2.7}$$

Theorem concerning the differential equation (2.6).

Hypotheses

i) $J(x) = \{\delta_{ij}\lambda_i(x)\}$

ii) $\mathcal{S}(k)$ is a closed sector (including $x = 0$) in $\mathcal{N}$ with the property that there exists positive constants

$\sigma > 0$, C_h, and C_p such that for every s, $\mathcal{S}(k)$ contains points $x_s = x_s(k)$ and rectifiable paths $P_s(x)$ from x_s to every point x of $\mathcal{S}(k)$ such that for $|\rho| > \rho_o$ and for $s \notin \bar{k}$

$$\int_{x_s}^{x} |\exp \rho(h_{sk}(x) - h_{sk}(t))||P_{sr}(t,\rho)dt| < C_h|\rho|^{-2\sigma} \tag{2.8}$$

for $s \in \bar{k}$

$$\int_{x_s}^{x} |P_{sr}(t,\rho)dt| < C_p \tag{2.9}$$

where σ, C_h, and C_p do <u>not</u> depend upon x.

iii) $QPQ = 0$, i.e.

$$P_{sr}(x,\rho) = 0 \quad \text{if } s \in \bar{k} \text{ and } r \in \bar{k} \tag{2.10}$$

<u>Conclusions</u>

i) To every k there exists a solution of (2.6), which we shall denote by $u(x,\rho)$, which is asymptotic, uniformly in $\mathcal{S}(k)$, to column k of the matrix $G(x,\rho)$ in (2.5), i.e.,

$$\max_s \max_{x \text{ in } \mathcal{S}(k)} |\delta_{sk} - u_s(x,\rho) \exp(-\rho h_k(x))| \to 0 \quad \text{as } |\rho| \to \infty$$

ii) The above solution $u(x,\rho)$ can be represented as the product of $\exp \rho h_k(x)$ and an expansion which converges uniformly for $|\rho| > \rho_o$, ρ_o defined below by (2.26), and x in $\mathcal{S}(k)$ (which <u>includes</u> $x = 0$). This representation is

$$u(x,\rho) = [\exp \rho h_k(x)][\ + \sum_{r=1}^{\infty} \omega^{(r)}(x,\rho)] \tag{2.11}$$

where $\omega_s(x,\rho) = \delta_{ks}$

$\omega^{(r)} = F^r\omega$

and the operator F^r is defined by

$$F^r\omega = F(F^{r-1}\omega) , \quad 2 \le r < \infty$$

and for r = 1

$$(2.12) \quad (Fv)_s = \int_{x_s}^{x} [\exp \rho(h_{sk}(x) - h_{sk}(t))] \sum_{r=1}^{n} P_{sr}(t,\rho)v_r(t,\rho)dt$$

(One should note that $\omega \exp \rho h_k(x)$ is precisely column k of $G(x,\rho)$).

iii) In a modified local norm the expansion (2.11) is asymptotic, uniformly for x in $\mathcal{S}(k)$, according to integral powers of $\rho^{-\sigma}$, i.e.,

$$(2.13) \quad \max_{x \text{ in } \mathcal{S}(k)} \left| \sum_{r=N}^{\infty} F^r\omega \right|_{mod} < C_N \rho^{-(N+1)\sigma}$$

where

$$(2.14) \quad |u(x,\rho)|_{mod} = \sum_{s \varepsilon \bar{k}} |u_s(x,\rho)| + \rho^{\sigma} \sum_{s \not\varepsilon \bar{k}} |u_s(x,\rho)|$$

In particular, the evaluation of (2.11) at x = 0,

$$(2.15) \quad u(0,\rho) = \omega + \sum_{r=1}^{\infty} (F^r\omega)(0,\rho)$$

is an expansion which is asymptotic in the modified norm.

Proof

Let

$$(2.16) \quad v(x,\rho) = u(x,\rho) \exp(-\rho h_k(x)) .$$

Then the integral equation for $v(x,\rho)$ is

$$v = \omega + Fv \tag{2.17}$$

where $\omega_s = \delta_{ks}$

and Fv is defined by (2.12).

To present a more detailed view of (2.12) we make use of (2.10) and note that $h_{sk}(x) = 0$ for $s \in \bar{k}$. Accordingly,

$$(Fv)_s = \begin{cases} \int_{x_s}^{x} \sum_{r}^{r\notin\bar{k}} P_{sr}(t,\rho)v_r(t,\rho)dt , & \text{if } s \in \bar{k} \\ \int_{x_s}^{x} [\exp \rho(h_{sk}(x)-h_{sk}(t))] \sum_{r=1}^{n} P_{sr}(t,\rho)v_r(t,\rho)dt & \\ & \text{if } s \notin \bar{k} . \end{cases} \tag{2.18}$$

Incidentally, following the suggestion of L. Nirenberg, the need for a modified norm obviated if operator F is modified to be $\hat{F}$ where

$$(\hat{F}v)_s = \begin{cases} \int_{x_s}^{x} \sum_{r}^{r\notin\bar{k}} P_{sr}(t,\rho)(Fv)_r \, dt , & \text{if } s \in \bar{k} \\ (Fv)_s \text{ of (2.18)} , & \text{if } s \notin \bar{k} \end{cases}$$

The integral equation for v is

$$v = \omega + \hat{F}v$$

but now the iterant solution converges in the customary maximum norm.

Let us define a global norm for a scalar as follows

$$\|v_r(x,\rho)\| = \max_{x \text{ in } \mathcal{S}(k)} |v_r(x,\rho)| \tag{2.19}$$

Then, with the aid of (2.19), (2.18) and (2.9) one readily obtains

$$\|(Fv)_s\| \le C_p \sum_{r}^{r \not\in \bar{k}} \|v_r\| \quad \text{if} \quad s \in \bar{k} \tag{2.20a}$$

Similarly, with the aid of (2.19), (2.18) and (2.8) one obtains

$$\|(Fv)_s\| \le C_h \rho^{-2\sigma} \sum_{r=\bar{1}}^{n} \|v_r\| \quad \text{if} \quad s \not\in \bar{k} \tag{2.20b}$$

Define a modified global norm for a vector v by

$$\|v\|_{mod} = \sum_{s}^{s \in \bar{k}} \|v_s\| + \rho^{\gamma} \sum_{s}^{s \not\in \bar{k}} \|v_s\| \tag{2.21}$$

with γ as yet unspecified. Clearly $\|\omega\|_{mod}$ is finite for any choice of γ. Indeed, $\|\omega\|_{mod} = 1$ independent of γ.

Let m_k denote the multiplicity of the eigenvalue $\lambda_k(x)$, i.e., m_k denotes the number of distinct integers in the class $\bar{k}$. Then (2.21) is combined with (2.20) to give

$$\|Fv\|_{mod} \le m_k C_P \sum_{r}^{r \not\in \bar{k}} \|v_r\| + (n-m_k) C_h \rho^{\gamma-2\sigma} \sum_{r=\bar{1}}^{n} \|v_r\|$$

which can be rewritten as

$$\|Fv\|_{mod} \le \rho^{\gamma-2\sigma} \left[C_1 \sum_{r}^{r \in \bar{k}} \|v_r\| + (C_1 \rho^{-\gamma} + C_2 \rho^{2\sigma-2\gamma}) \rho^{\gamma} \sum_{r}^{r \not\in \bar{k}} \|v_r\| \right]$$

where

$$C_1 = (n-m_k) C_h \tag{2.22a}$$

and

$$C_2 = m_k C_P \tag{2.22b}$$

In particular, let us select

(2.23) $$\gamma = \sigma$$

and define

(2.24) $$C_F = \max\ (C_1\rho_o^{-\sigma} + C_2, C_1)$$

then

(2.25) $$\|Fv\|_{mod} \leq C_F\rho^{-\sigma}\ \|v\|_{mod}$$

for $\rho > \rho_o$.

Select ρ_o such that

(2.26) $$C_F\rho_o^{-\sigma} < 1$$

Then for $\rho > \rho_o$ the iterations of $v = \omega + Fv$ are dominated in $\|\ \|_{mod}$ by the convergent geometric series $\sum_{j=0}^{\infty} (C_F\rho_o^{-\sigma})^j$. In view of (2.26) these iterations converge and conclusion (ii) is verified. Since the iterations converge

$$\|v - \sum_{r=0}^{N} (F^r\omega)\|_{mod} \leq \frac{\|F^{r+1}\omega\|_{mod}}{1 - C_F\rho_o^{-\sigma}}$$

But repeated application of (2.25) shows that

$$\|F^{r+1}\omega\|_{mod} \leq C_F^{r+1}\ \rho^{-(r+1)\sigma}$$

so that v is asymptotic to $\omega + \sum_{r=0}^{\infty} F^r\omega$ in $\|\ \|_{mod}$. Moreover,

$$\max_{x \text{ in } \mathcal{L}(k)} |u|_{mod} \leq \|u\|_{mod}$$

so that conclusion (iii) is proved. Conclusion (i) is a simple consequence of (iii). Q.E.D.

Better perspective on hypothesis (iii), i.e., (2.10), is furnished by the following lemma.

Lemma 1. In order that (2.6) be transformed by

$$u = Ty \tag{2.27}$$

into

$$\frac{d}{dx} y = [\rho J(x) + \hat{P}(x,\rho)]y$$

with

$$Q\hat{P}Q = 0$$

it is sufficient that T be defined by

$$T = QTQ + \overline{Q} \tag{2.28}$$

and

$$\frac{d}{dx} T = QPQT \tag{2.29}$$

with the non-zero columns of QTQ independent.

The proof of this lemma proceeds by a direct verification. For example, the existence of T^{-1} follows from the definition of T and the fact that

$$T^{-1}JT = J \tag{2.30}$$

follows from (2.28) and the similar direct sum nature of J, i.e.,

$$J = \lambda_k Q + \overline{Q}J\overline{Q} \tag{2.31}$$

Thus we see that hypothesis (iii) is inessential and, indeed, it can be eliminated by incorporating Lemma 1 into the preceding theorem.

In order to simplify the "connection to the turning" one can take the identity matrix as the initial value for T at x = 0, i.e.,

(2.32) $$T = Q + \bar{Q} \qquad \text{at } x = 0 .$$

Of course T still occurs in the asymptotic expansion of u and it is often desirable to seek the simplest T. For this purpose one should note that all the elements of P need not partake in the definition of T. Suppose

(2.33) $$P = P^0 + P^1$$

and (2.9) is replaced by

$$\left.\begin{aligned} &(2.34a) \quad \int_{x_s}^{x} |(P^1)_{sr}dt| < C_p'\rho^{-\sigma} \\ &(2.34b) \quad \int_{x_s}^{x} |(P^0)_{sr}dt| < C_p \end{aligned}\right\} \text{ if } s \in \bar{k}$$

and (2.10) is replaced by

(2.35) $$QP^0Q = 0 .$$

Then the preceding theorem will apply after obvious modifications. For example, the change in (2.24) is that C_1 will be replaced by $C_1 + m_k C_p'$. Consequently it is sufficient to define T by (2.28) and

(2.36) $$\frac{d}{dx} T = QP^0QT$$

The last item which remains to be done is to delimit a sector $\mathcal{A}(k)$ in which (2.8) is valid. This sector depends almost entirely on the contours

$$\text{Re } \rho h_{sk}(x) = \text{constant} .$$

The path from x_s to x is to be one along which Re $\rho h_{sk}(x)$ decreases. In order that x = 0 be on such a path, or, as we shall say, in order that x = 0 be accessible from x_s, it is necessary that either

$$\text{Re } h_{sk}(x_s) > 0$$

or

$$x_s = 0$$

The contours

$$\text{Re } \rho h_{sk}(x) = 0$$

are called transition lines since as one crosses this line the relative dominance between exp $\rho h_k(x)$ and exp $\rho h_s(x)$ alters. These transition lines issue from x = 0 since by (1.6)

$$h_s(0) = h_k(0) = 0 \ .$$

When $h_{sk}(x)$ has a zero of order r (r an integer) at x = 0 then the transition lines partition the neighborhood of x = 0 into 2r so-called minimum sectors. For a fixed x_s the x which are accessible forms a sector at x = 0 of at most three adjacent minimum sectors (Fowler [1], Friedrichs [1,§18]).

The estimate in (2.8) can be made as follows. Introduce the path length, ℓ, measured from x as a new variable of integration, i.e.,

$$\ell = \ell(t) \quad \text{with} \quad \ell(x) = 0 \quad \text{and} \quad \ell(x_s) = L$$

where L is the total path length. A sufficient condition for (2.8) is that there exist constants k_1 with Re $k_1 > 0$, $r > 1$, $k_2 > 0$, $\theta < 1$, such that for all t on the path from x_s to x

(2.36) $$\text{Re } \rho(h_{sk}(x) - h_{sk}(t)) \leq -|\rho| \text{ Re } k_1 \ell^r$$

(2.37) $$|P_{rs}(t,\rho) \frac{dt}{d}| < k_2 \ell^{-\theta}$$

The estimate in (2.8) is then valid with

$$\sigma = \frac{1 - \theta}{r}$$

$$C_h = k_2 \int_0^\infty |\exp(-k_1 \ell^r)| |\ell^{-\theta} d\ell|$$

(when $L = \infty$ it is often useful to note that in the range where ℓ is boundedly distant from zero it is sufficient to replace $\ell^{-\theta}$ in (2.37) by, say, $\exp k_1 \ell^r$.)

In most circumstances, if a point x is at all accessible from x_s then it is accessible along a path on which (2.36) and (2.37) and, hence, (2.8) is valid.

III. The Second Order Equation

A. Introduction

In this chapter the turning point problem (see Chapter I) shall be solved (in conjunction with Chapter II) for certain equations of the form

(3.1) $$\frac{d^2u_1}{dx^2} = [\rho^2 a(x) + \rho b(x) + c(x,\rho)]u \ .$$

where ρ is a large real parameter

x ranges over a finite neighborhood surrounding the turning point

$a(x)$, $b(x)$ and $c(x,\rho)$ are regular in x

$c(x,\rho)$ is uniformly bounded.

For convenience, the turning point shall be at $x = 0$. Denote the order of the zero of $a(x)$ and $b(x)$ at $x = 0$ by, respectively, α and β.

In connection with (3.1) it is natural to consider the (related) equation

$$\frac{d^2}{dx^2} g_1 = \rho^2 x^\alpha g_1 \ . \tag{3.2}$$

The main point of this chapter is to indicate a method by which the above two equations can be related whenever

$$\beta > \frac{\alpha}{2} - 1 \ . \tag{3.3}$$

The situation where $a(x)$ is identically zero, i.e., $\alpha = \infty$, will be excluded. This involves no loss of generality for $\sigma = \rho^{1/2}$ can serve equally well as the large real parameter. In terms of σ (3.1) is

$$\frac{d^2}{dx^2} u_1 = [\sigma^2 b(x) + c(x,\sigma^2)]u_1$$

and since a turning point occurs at $x = 0$, $b(x)$ is <u>not</u> identically zero.

The restriction (3.3) is reasonable from the following heuristic considerations. The formal asymptotic solutions of (3.1) and (3.2) are, respectively,

$$u_1 \sim \left\{[a(x)]^{-\frac{1}{4}} \exp \rho \int^x [a(z)]^{\frac{1}{2}} dz \right\} \exp \tfrac{1}{2} \int^x [a(z)]^{-\frac{1}{2}} b(z)\, dz$$

and

$$g_1 \sim [x^\alpha]^{-\frac{1}{4}} \exp \rho \int^x [z^\alpha]^{\frac{1}{2}} dz$$

These asymptotic forms resemble each other down to the turning point provided $[a(x)]^{-1/2}b(x)$ is integrable at $x = 0$ and (3.3) is a sufficient condition for this integrability.

An alternate heuristic procedure is to first rewrite (3.1) in terms of a stretched variable t (Friedrichs [2]) where

$$t = \rho^{\varrho}x \quad , \quad \varrho > 0$$

to obtain

$$\frac{d^2}{dt^2} u_1 = (\rho^{2-2\varrho-a\varrho} a_o t^a + \rho^{1-2\varrho-\beta\varrho} b_o t^\beta + \ldots)u_1$$

This resembles (3.2) provided the term t^a is dominant for all possible stretching, i.e.,

$$2 - 2\varrho - a\varrho = 0 \quad \text{implies} \quad 1 - 2\varrho - \beta\varrho < 0$$

and this condition is precisely (3.3). Thus (3.3) may be described as the non-transitional case in that a(x) is the dominant term both when x is boundedly distant from zero as well as when x is infinitesimally close to zero.

For the sake of clarity and for generality the analysis is carried out in terms of a system of the form

(3.4a) $$\frac{d}{dx} u = [\rho A(x) + B(x) + \rho^{-1}C(x,\rho)]u$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$A(x) = \begin{bmatrix} 0 & 1 \\ a(x) & 0 \end{bmatrix}$$

Non-zero entries in $C(x,\rho)$ will be admissible provided that each entry is uniformly bounded. Similarly, none of the entries in $B(x)$ need be identically zero provided only that each entry has a zero at $x = 0$ of order greater than or equal to β where β is constrained by (3.3).

The related system for (3.4) is the matrix equation

$$\frac{d}{dx} G = \rho A(x) G \tag{3.5a}$$

where

$$A(x) = \begin{bmatrix} 0 & 1 \\ x^{\alpha} & 0 \end{bmatrix} \tag{3.5b}$$

In order to relate (3.4) and (3.5) one must first either transform (3.4a) so that $a(x)$ of (3.4b) becomes x^{α} or else one must transform (3.5a) so that x^{α} of (3.5b) becomes $a(x)$. For convenience, the former course is selected here and detailed in section B. The system which results, i.e., (3.4) with $a(x)$ replaced by x^{α}, is called a canonical system.

In section C we will exhibit solutions to a generalization of (3.5) which serve to reduce a corresponding generalization of the above canonical system to the form where $A(x)$ is diagonal, $B(x)$ is integrable, and $\rho^{-1}C(x,\rho)$ is contracting. The further discussion of this latter system proceeds as in Chapter II.

B. Reduction to the Canonical System

This section shall present in matrix form the transformation exploited by Langer [1].

Let

$$x = x(\tilde{x}) \ , \quad u = P_o v$$

where

$$P_o = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{x}' \end{bmatrix}$$

' denotes differentiation with respect to x.

When u satisfies (3.4) the equation for v is

$$(3.6)\quad \tilde{x}' \frac{d}{d\tilde{x}} v = \left[\rho \begin{bmatrix} 0 & \tilde{x}' \\ \frac{a(x)}{\tilde{x}'} & 0 \end{bmatrix} - \frac{\tilde{x}''}{\tilde{x}'} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + P_o^{-1} B P_o + \rho^{-1} P_o^{-1} C P_o \right] v$$

The determination of $\tilde{x}(x)$ follows from the requirement that

$$a(x) = \tilde{x}^{a}(\tilde{x}')^2$$

i.e.,

$$(3.7)\qquad \tilde{x}(x) = \left[\frac{p}{2} \int_o^x [a(z)]^{\frac{1}{2}} \, dz \right]^{\frac{2}{p}}$$

where $p = a + 2$.

Of course, the implication is that $\tilde{x} = \tilde{x}(x)$ is a 1-1 regular transformation at $x = 0$. This is, indeed, the case since near $x = 0$, $\tilde{x}(x)$ approaches

$$\left[\frac{p}{2} \int_o^x z^{\frac{a}{2}} \, dz \right]^{\frac{2}{p}}$$

and this is precisely equal to x.

It may happen that the zero of $\tilde{x}''$ is greater than $\frac{a}{2} - 1$ so that (3.7) is already in the desired form. If this is not the case then it turns out that the $\tilde{x}''/\tilde{x}'$ term of (3.6) can be removed at no expense to A(x) or B(x) by

$$(3.8)\qquad v = \begin{bmatrix} \phi & 0 \\ \rho^{-1}\phi' & \phi \end{bmatrix} w$$

where $\phi^{2\tilde{x}'} = 1$. Consequently, there is no loss of generality if in (3.4) $a(x)$ is taken to be x^{a}.

C. Diagonalization of the Canonical System

In this section we shall allow ourselves the luxury of discussing a canonical system which is more general than is needed for (3.1). This system is

$$\frac{d}{dx} u = [\rho A(x) + B(x) + \rho^{-1} C(x,\rho)]u \tag{3.9}$$

where

$$A(x) = \begin{bmatrix} 0 & x^{a_1} \\ x^{a_2} & 0 \end{bmatrix} \tag{3.10}$$

a_1 and a_2 are positive integers

$B(x)$ has a zero of order β at $x = 0$

with

$$\beta > \frac{|a_1 - a_2|}{2} - 1 \tag{3.11}$$

and $C(x,\rho)$ is uniformly bounded.

Incidentally, under a transformation of the form $x = z^{k}$ where k is a positive integer, a_1, a_2, and β change in such a manner that (3.11) is invariant.

We first analyze the matrix equation

$$\frac{d}{dx} G = \rho A(x) G \ . \tag{3.12}$$

For $x \neq 0$, the Jordan form of $A(x)$ is a diagonal matrix which we denote by $J(x)$ and we define E by

$$\frac{d}{dx} E = \rho J E \tag{3.13}$$

Our goal is the "diagonalization" of (3.9) according to the strategy outlined in Chapter I. For this purpose it is sufficient to construct a solution G to (3.12) such that for x in a sector in the x-plane and with V defined by

$$V = GE^{-1} \tag{3.14}$$

the following three conditions are satisfied.

$$\begin{cases} \text{a)} & V^{-1}BV \text{ is integrable} \\ \text{b)} & \rho^{-1}V^{-1}CV \text{ is contracting} \\ \text{c)} & V \text{ is known asymptotically at } x = 0. \end{cases} \tag{3.15}$$

With

$$u = Vw \tag{3.16}$$

the equation for w is

$$\frac{d}{dx} w = [\rho J + V^{-1}BV + \rho^{-1}V^{-1}CV]w \tag{3.17}$$

and when (3.15) prevails the further discussion of (3.17) proceeds as in Chapter II.

Condition (3.15c) results from the fact that V can be written explicitly in terms of Hankel functions. Indeed, conditions (3.15a) and (3.15b) could be demonstrated from the representations of the Hankel functions as definite integrals. However, this integral representation will be ignored so that the present discussion can be applied almost directly to certain systems of order n.

The parameter ρ is suppressed from (3.12) and (3.13) when one goes into the plane of the stretched independent variable t where

(3.18)
$$t = \rho^{\frac{2}{p}} x$$
$$p = a_1 + a_2 + 2$$

Define

$$G = \begin{bmatrix} \rho^{\frac{a_2-a_1}{2p}} & 0 \\ 0 & \rho^{\frac{a_1-a_2}{2p}} \end{bmatrix} \hat{G}$$

The so-called stretched equation is then

(3.20)
$$\frac{d}{dt}\hat{G} = A(t)\hat{G}$$

where A is defined by (3.10). In terms of t the solution of (3.13) is

(3.21)
$$E = \exp \frac{2}{p} t^{\frac{p}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A line along which Re $t^{p/2} = 0$ is called a transition line (see, for example, Friedrichs [1,§19]). If $a_1 + a_2$ is not an even integer we have to consider a doubly sheeted t-plane or what amounts to the same thing a singly sheeted $\hat{t}$-plane where $t = \hat{t}^2$. The variable x would then correspond to any one sheet of the t-plane. However, in order to simplify the discussion, we shall suppose that the integer $a_1 + a_2$ is even. There are then p distinct transition lines and these occur at

(3.22)
$$\arg t = \frac{\pi h}{p}, \quad h = 1,3,\ldots,2p-1 .$$

Consider the set of p closed sectors each of angle less than $\frac{2\pi}{p}$ and each of which is bisected by a different transition line. These sectors form an overlapping covering of the turning point.

In this chapter the discussion shall be restricted to a typical such sector and this sector shall be denoted by $\mathcal{S}$. Of course, one should keep in mind that x is to be in a finite neighborhood of x = 0 so that in $\mathcal{S}$, $|t|$ will not grow more rapidly than $\rho^{2/p}$. It is easily seen, see Appendix A, that

$$(3.23)\qquad \hat{G} = \begin{bmatrix} t^{\frac{a_1+1}{2}} & \\ 0 & \left(\frac{pt}{2}\right)^{\frac{a_2+1}{2}} \end{bmatrix} \begin{bmatrix} H^{(1)}_{(\frac{a_1+1}{p})}(i\zeta) & H^{(2)}_{(\frac{a_1+1}{p})}(i\zeta) \\ H^{(1)}_{-(\frac{a_2+1}{p})}(i\zeta) & H^{(2)}_{-(\frac{a_2+1}{p})}(i\zeta) \end{bmatrix} K$$

where $\zeta = \frac{2}{p}\, t^{p/2}$ and K is a constant matrix, is a solution to (3.20). (Of course, K will be selected so as to have a non-vanishing determinant.) Define $\hat{G}$ by (3.23) (with K not yet specified), G by (3.19), E by (3.21), and V by (3.14). For each sector $\mathcal{S}$ it is a simple matter to define K in (3.23) so as to obtain the following two estimates for V,

$$(3.24)\qquad V < \text{const } x^{-\left|\frac{a_1-a_2}{2}\right|}$$

and

$$(3.25)\qquad V \quad \text{const } \rho^{\left|\frac{a_1-a_2}{2p}\right|}$$

Let us accept these estimates for the moment. In order to compute V^{-1} we first note that the determinant of V, denoted by det V, is a non-zero constant independent of x and ρ since

$$\det V = \det G \quad \text{(by (3.14) and (3.21))}$$
$$= \det \hat{G} \quad \text{(by (3.19))}$$
$$= \text{constant} \quad \text{(by (3.20) since the trace of } A(t) = 0)$$
$$\neq 0 \quad \text{(by (3.23) since the columns of } G \text{ are independent)}$$

Consequently, the elements of V^{-1} are a permutation of the elements of V so that estimates analogous to (3.24) and (3.25) are valid for V^{-1} and finally

$$V^{-1}BV < \text{const } x^{\beta - \left|\frac{\alpha_1 - \alpha_2}{2}\right|} \tag{3.26}$$

and

$$\rho^{-1}V^{-1}CV < \text{const } \rho^{\left|\frac{\alpha_1 - \alpha_2}{p}\right| - 1} \tag{3.27}$$

This already verifies (3.15) since by (3.11)

$$\beta - \left|\frac{\alpha_1 - \alpha_2}{2}\right| > -1$$

and by (3.18)

$$\left|\frac{\alpha_1 - \alpha_2}{p}\right| - 1 < 0$$

The only item which remains to be completed is the derivation of the estimates (3.24) and (3.25).

The behavior of $\hat{G}$ in a finite neighborhood of t = 0 is already described by (3.20). In order to estimate the behavior of $\hat{G}$ outside a finite neighborhood of t = 0 let

$$\hat{G} = \begin{bmatrix} t^{\frac{\alpha_1 - \alpha_2}{4}} & 0 \\ 0 & t^{\frac{\alpha_2 - \alpha_1}{4}} \end{bmatrix} Y \tag{3.28}$$

and

$$Y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} T \tag{3.29}$$

Corresponding to (3.20) for $\hat{G}$, the equation for Y is

$$\frac{d}{dt} Y = \left[t^{\frac{a_1+a_2}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{a_1-a_2}{4t} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right] Y$$

and the equation for T is

$$\frac{d}{dt} T = \left[t^{\frac{a_1+a_2}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{a_1-a_2}{2t} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] T \tag{3.30}$$

From well known theory (see, for example, Friedrichs [1,§13]) (3.30) has a solution T whose asymptotic form for t in Δ but t large is a constant multiplied by the matrix E of (3.21). Select K in (3.23) so that this T is the solution of (3.30) which corresponds to (3.23). One should note that K can be determined explicitly since the continuation problem for the Bessel equation has already been solved. (Watson [1,Chap.7]).

From the above selection of K and by (3.28) and (3.29)

$$\hat{G}E^{-1} < \text{const} \begin{bmatrix} t^{\frac{a_1-a_2}{4}} & 0 \\ 0 & t^{\frac{a_2-a_1}{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{for } |t| \geq t_o \tag{3.31a}$$

and by (3.20)

$$\hat{G}E^{-1} < \text{const}, \quad \text{for } |t| \leq t_o \tag{3.31b}$$

where t_o is a fixed, but sufficiently large, real number.

For definiteness, let $a_2-a_1 \geq 0$. Then, from (3.31)

$$\hat{G}E^{-1} < \text{const} \begin{bmatrix} 1 & 0 \\ 0 & 1+t^{\frac{a_2-a_1}{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S} \tag{3.32}$$

or, since by (3.18)

$$\rho^{\frac{a_1-a_2}{2p}} t^{\frac{a_2-a_1}{4}} = x^{\frac{a_2-a_1}{4}} \tag{3.33}$$

(3.19) is combined with (3.32) to form

$$GE^{-1} < \text{const} \begin{bmatrix} \rho^{\frac{a_2-a_1}{2p}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S} \tag{3.34}$$

which, with (3.14), results in (3.25).

On the other hand, from (3.31)

$$\hat{G}E^{-1} < \text{const} \begin{bmatrix} t^{\frac{a_1-a_2}{4}} & 0 \\ 0 & 1+t^{\frac{a_2-a_1}{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S} \tag{3.35}$$

and this is combined with (3.33) and (3.19) to form

$$GE^{-1} < \text{const} \begin{bmatrix} x^{\frac{a_1-a_2}{4}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.36}$$

which, with (3.14), results in (3.24).

IV. A Fourth Order Equation

In this chapter we shall indicate how one can obtain connection formula for an equation of the form

$$(4.1)\quad u^{(iv)} - \rho^2\{[a(x)+\rho^{-1}b_o(x)+\rho^{-2}c_o(x,\rho)]u^{(ii)} + [b_1(x)+\rho^{-2}c_1(x,\rho)]u^{(i)} + (b_2(x)+\rho^{-1}c_2(x,\rho)]u\} = 0$$

in terms of connection formula for

$$(4.2)\qquad g^{(iv)} - \rho^2 a(x) g^{(ii)} = 0$$

when at $x = 0$, $a(x)$ has precisely a simple zero while each of the terms $b_o(x)$, $b_1(x)$, and $b_2(x)$ has at least a simple zero. The terms $c_o(x,\rho)$, $c_1(x,\rho)$, and $c_2(x,\rho)$ represent terms which are bounded uniformly in x and ρ. Equation (4.1) differs from a classical equation which arises in the theory of hydrodynamic stability in the above requirement that $b_2(0) = 0$ and (4.2) is correspondingly simpler than the customary related equation (Wasow [1], Lin [1]).

Equation (4.1) is the first component equation of the system

$$(4.2)\quad \frac{d}{dx}u = \left[\rho\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&a(x)&0\end{bmatrix} + \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\b_2(x)&b_1(x)&b_0(x)&0\end{bmatrix} + \rho^{-1}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\c_2&\rho^{-1}c_1&c_0&0\end{bmatrix}\right]u$$

It is a routine matter to reduce (4.1) or (4.2) to an equation of the same form but where a(x) is replaced by x. (See for example, Chapter III,§B.) For simplicity, this will be assumed to have been done and we will start with the equation

$$(4.3)\qquad \frac{d}{dx}u = \left[\rho\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&x&0\end{bmatrix} + B(x) + \rho^{-1}C(x,\rho)\right]u$$

None of the entries in $B(x)$ or $C(x,\rho)$ need be identically zero for the subsequent analysis. The only restriction on $B(x)$ is that the entries in the last row vanish at $x = 0$. Each entry in $C(x,\rho)$ is required to be bounded uniformly in x and ρ and, moreover, $\rho^{1/3}$ times each entry in the second column of $C(x,\rho)$ is to be uniformly bounded.

The related system for (4.3) is the <u>matrix</u> equation

$$(4.4)\qquad \frac{d}{dx}H = \rho\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&x&0\end{bmatrix}H$$

and this, in turn is related in an obvious manner to the system

$$(4.5)\qquad \frac{d}{dx}G = \rho\begin{bmatrix}0&1&0\\0&0&1\\0&x&0\end{bmatrix}G$$

namely,

$$(4.6)\qquad H = \left[\begin{array}{c|ccc}1&0&0&0\\0&&&\\0&&G&\\0&&&\end{array}\right]$$

From (4.5) one obtains the parameter free stretched system

$$\frac{d}{dt}\hat{G} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & t & 0 \end{bmatrix}\hat{G} \tag{4.7}$$

by the definitions

$$G = \begin{bmatrix} \rho^{\frac{1}{2}} & 0 & 0 \\ 0 & \rho^{\frac{1}{6}} & 0 \\ 0 & 0 & \rho^{-\frac{1}{6}} \end{bmatrix}\hat{G} \tag{4.8}$$

and

$$t = \rho^{2/3}x \tag{4.9}$$

The solution to (4.7) can be written explicitly in terms of Airy functions Ai(t) and Bi(t), namely,

$$\hat{G} = \begin{bmatrix} \rho^{-\frac{1}{2}} & \int_0^t \mathrm{Ai}(z)dz & -\int_0^t \mathrm{Bi}(z)dz \\ 0 & \mathrm{Ai}(t) & -\mathrm{Bi}(t) \\ 0 & \frac{d}{dt}\mathrm{Ai}(t) & -\frac{d}{dt}\mathrm{Bi}(t) \end{bmatrix} \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{K}} \\ 0 & & \end{array}\right] \tag{4.10}$$

where K is a constant matrix (not yet specified).

It is clear from (4.7) that $\hat{G}$ of (4.10) is bounded in a finite neighborhood of t = 0. In order to estimate $\hat{G}$ for t large let

$$\hat{G} = \begin{bmatrix} t^{-\frac{3}{4}} & 0 & 0 \\ 0 & t^{-\frac{1}{4}} & 0 \\ 0 & 0 & t^{\frac{1}{4}} \end{bmatrix} Y \tag{4.11}$$

The equation for Y is then

$$\frac{d}{dt} Y = \left[t^{\frac{1}{2}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4t} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right] Y$$

Finally, let

$$Y = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} T \tag{4.12}$$

to obtain

$$\frac{d}{dt} T = \left[t^{\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \frac{1}{8t} \begin{bmatrix} 6 & 5 & 5 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} \right] T \tag{4.13}$$

The first column of T is selected as

$$\rho^{-\frac{1}{2}} t^{\frac{3}{4}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so as to be consistent with (4.10). Define E as

$$E = \frac{2}{3} t^{\frac{3}{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{4.14}$$

A transition line is defined as a line along which $\operatorname{Re} t^{3/2} = 0$. Let $\mathcal{S}$ be a closed sector which contains one and only one transition line.

For t large (4.13) has a solution T whose last two columns are asymptotic to a constant multiplied by the last two columns of E. Select K in (4.10) so that this T is the solution to (4.13) which corresponds to (4.10).

Let $[\hat{G}E^{-1}]$ denote the last two columns of $\hat{G}E^{-1}$. Then in $\mathcal{S}$

$$(4.15a)\quad [\hat{GE}^{-1}] < \text{const} \begin{bmatrix} t^{-\frac{3}{4}} & 0 & 0 \\ 0 & t^{-\frac{1}{4}} & 0 \\ 0 & 0 & t^{\frac{1}{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad t_o \leq |t|$$

$$(4.15b)\quad [\hat{GE}^{-1}] < \text{const}, \quad |t| \leq t_o$$

where t_o is a fixed, but sufficiently large, real number. The first global estimate for GE^{-1} which follows from (4.15) is

$$(4.16)\quad [\hat{GE}^{-1}] < \text{const} \begin{bmatrix} t^{-\frac{3}{4}} & 0 & 0 \\ 0 & t^{-\frac{1}{4}} & 0 \\ 0 & 0 & t^{\frac{1}{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S}$$

i.e., from (4.8) and (4.9)

$$(4.17)\quad [GE^{-1}] < \text{const} \begin{bmatrix} x^{-\frac{3}{4}} & 0 & 0 \\ 0 & x^{-\frac{1}{4}} & 0 \\ 0 & 0 & x^{\frac{1}{4}+\rho^{-\frac{1}{6}}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S}$$

The second global estimate which follows from (4.15) is

$$(4.18)\quad [\hat{GE}^{-1}] < \text{const} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{\frac{1}{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S}$$

i.e., from (4.8) and (4.9)

$$(4.19)\quad [GE^{-1}] < \text{const} \begin{bmatrix} \rho^{\frac{1}{2}} & 0 & 0 \\ 0 & \rho^{\frac{1}{6}} & 0 \\ 0 & 0 & x^{\frac{1}{4}} + \rho^{-\frac{1}{6}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{all } t \text{ in } \mathcal{S}$$

In order to carry over these results to H of (4.4) define

$$\bar{E} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & E & \\ 0 & & & \end{array}\right]$$

where E is defined by (4.14), and define

$$(4.21)\qquad V = H\bar{E}^{-1} .$$

Then (4.17) results in the estimates

$$(4.22)\qquad V < \text{const} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x^{-\frac{3}{4}} & x^{-\frac{3}{4}} \\ 0 & 0 & x^{-\frac{1}{4}} & -x^{-\frac{1}{4}} \\ 0 & 0 & 1+x^{\frac{1}{4}} & 1+x^{\frac{1}{4}} \end{bmatrix}$$

while (4.19) results in

$$(4.23)\qquad V < \text{const} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \rho^{\frac{1}{2}} & \rho^{\frac{1}{2}} \\ 0 & 0 & \rho^{\frac{1}{6}} & -\rho^{\frac{1}{6}} \\ 0 & 0 & \rho^{-\frac{1}{6}} + x^{\frac{1}{4}} & \rho^{-\frac{1}{6}} + x^{\frac{1}{4}} \end{bmatrix}$$

In order to estimate V^{-1} one first notes that the determinant of V, denoted by det V, is a non zero constant (independent of x and ρ) since

$$\begin{aligned}
\det V &= \det H && \text{(by (4.14) and (4.21))}\\
&= \det G && \text{(by (4.6))}\\
&= \rho^{1/2} \det \hat{G} && \text{(by (4.8))}\\
&= (\det K)\ \text{Wronskian of } (A_i(t), B_i(t)) && \text{(by (4.10))}\\
&= \text{constant} \neq 0 \ .
\end{aligned}$$

Consequently, the estimate for V^{-1} is the same as the estimate for the matrix which is obtained from V by first replacing each element by its cofactor and then interchanging rows and columns. Thus, from (4.22), one estimates

$$(4.24) \qquad V^{-1} < \text{const} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -x^{-1} \\ 0 & 0 & 1+x^{\frac{1}{4}} & x^{-\frac{1}{4}} \\ 0 & 0 & 1-x^{\frac{1}{4}} & x^{-\frac{1}{4}} \end{bmatrix}$$

while from (4.23)

$$(4.25) \qquad V^{-1} < \text{const} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\rho^{\frac{1}{3}} & -\rho^{\frac{2}{3}} \\ 0 & 0 & \rho^{-\frac{1}{6}} + x^{\frac{1}{4}} & -\rho^{-\frac{1}{6}} - x^{\frac{1}{4}} \\ 0 & 0 & \rho^{\frac{1}{6}} & \rho^{\frac{1}{6}} \end{bmatrix}$$

Define

$$(4.26) \qquad J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -t^{\frac{1}{2}} \end{bmatrix}$$

and

$$w = Vu \tag{4.27}$$

where V is defined by (4.21). Then

$$\frac{d}{dx}\, u = [\rho J + V^{-1}BV + \rho^{-1}V^{-1}CV]u \tag{4.28}$$

In the following we restrict ourselves to a finite neighborhood of $x = 0$. From (4.24) and the fact that the last row of B has a zero at $x = 0$ it follows that $V^{-1}B$ is bounded. This fact, in combination with (4.22) shows that

$$V^{-1}BV < x^{-\frac{3}{4}} \tag{4.29}$$

which is certainly integrable. Finally from (4.23) and the fact that $\rho^{1/3}$ times the second column of C is bounded it follows that

$$CV < \rho^{\frac{1}{6}}$$

This fact in combination with (4.25) shows that

$$\rho^{-1}V^{-1}CV < \rho^{-\frac{1}{6}} \tag{4.30}$$

which is certainly contracting.

The estimates (4.29) and (4.30) together with the obvious fact that J of (4.26) is diagonal show that the further discussion of (4.28) proceeds as in Chapter II.

V. Conclusions

The classical work on the turning point problem has relied upon a canonical system which is a perturbed diagonal system where the integral of the perturbation term is asymptotically small. A more appropriate foundation has been elaborated in this report, namely a canonical system which is a perturbed diagonal system where the integral of the perturbation term is merely bounded. In order to clarify the scope of this latter canonical system it is important to delimit the class of perturbations which are "accessible" with respect to a given comparison system. For example, if the comparison system is

$$\frac{d}{dx} G(x,\rho) = \rho A(x,\rho)G(x,\rho)$$

then the fact that the system

$$\frac{d}{dx} u(x,\rho) = [\rho A(x,\rho) + B(x,\rho)]u(x,\rho)$$

is reducible to the improved canonical system via solutions of the comparison system will imply a condition on $B(x,\infty)$ and a condition on $B(x,\rho)-B(x,\infty)$. The precise nature of these conditions have been indicated (in Chapters III and IV) when the comparison equation has the form of certain chain systems. The results in Chapter III on a second order equation merits special attention since it provides a good perspective on the many papers previously published on this problem (See, for example, Langer [3], Jeffreys [1]).

Appendix A: The (General) Hypergeometric Equation

An equation of the form

$$\text{(A-1)} \qquad \prod_{i=1}^{n} (\theta + \beta_i)y = \gamma t^{\kappa} \prod_{i=1}^{m} (\theta + \alpha_i)y$$

where

$$\theta = \frac{d}{dt}$$

is denoted as a (general) hypergeometric equation.

Related to this equation are the (general) hypergeometric series which are defined in the following notation, [Watson, p. 100].

$$\text{(A-2)} \qquad {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;t) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{t^n}{n!}$$

where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha n-1)$; $(\alpha)_0 = 1$. For this series the following lemma is readily verified.

Lemma 1: ${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;t)$ satisfies the equation

$$\text{(A-3)} \qquad \theta(\theta+\beta-1)\ldots(\theta+\beta_q-1)y = t(\theta+\alpha_1)\ldots(\theta+\alpha_p)y$$

Lemma 1 does not yet apply to (A-1) and in order to remedy this we state two more trivial lemmas:

Lemma 2: $f(\theta)t^j y = t^j f(\theta+j)y$ where $f(\theta)$ is a polynomial with constant coefficients.

Lemma 3: If $z = t^{\kappa}$ then $\kappa\theta_z = \theta_t$ where $\theta_t = t\frac{d}{dt}$ and $\theta_z = z\frac{d}{dz}$.

By means of these last two lemmas we now convert (A-1) into (A-3) as follows. First y of (A-1) is transformed according to

$$\text{(A-4)} \qquad y = t^{\beta_j} v$$

From Lemma 2 the resulting equation for v is

$$(A\text{-}5) \qquad \prod_{i=1}^{n} (\theta - \beta_i + \beta_j)v = \gamma t^{k} \prod_{i=1}^{m} (\theta + \alpha_i + \beta_j)v$$

Next, with the aid of Lemma 3, (A-5) is rewritten in terms of

$$(A\text{-}6) \qquad z = \frac{\gamma t^{k}}{k^{n-m}}$$

and the new equation is

$$(A\text{-}7) \qquad \prod_{i=1}^{n} (\theta_z + \frac{-\beta_i+\beta_j}{k})y = z \prod_{i=1}^{m} (\theta_z + \frac{\alpha_i+\beta_j}{k})$$

Finally, we define β_i for $i > n$ as

$$\beta_{i+n} = \beta_i$$

so that equation (A-7) can be rewritten as

$$(A\text{-}8) \qquad \theta_z \prod_{i=1}^{n-1} (\theta_z + \frac{-\beta_{j+i}+\beta_j}{k})y = z \prod_{i=1}^{m} (\theta_z + \frac{\alpha_i+\beta_j}{k})y$$

This equation is precisely in the form covered by Lemma 1 so that one can exhibit the following n solutions to (A-1).

$$(A\text{-}9) \qquad y_j = t^{\beta_j} {}_mF_{n-1}(\alpha_{1j},\ldots\alpha_{mj};\beta_{1j},\ldots\beta_{mj};z)$$

where $\alpha_{ij} = \frac{\alpha_i+\beta_j}{k}$ and $\beta_{ij} = \frac{-\beta_{j+i}+\beta_j}{k} + 1$, $j = 1,2,\ldots,n$.

These solutions will be distinct provided that none of the quantities β_j/k differ by zero or an integer. Moreover, (A-9) will represent actual solutions (and not just formal solutions) provided $n \geq m$. When $n > m$ then the series converge for all t.

One should note that the above results give a simple sufficient condition in order that n independent regular solutions exist at the singular point t = 0. This sufficient condition is that $n \geq m$, β_i be non-negative integers, and none of the β_i differ by a multiple of k.

Let us now show the hypergeometric character of the equation,

$$\text{(A-10)} \qquad \prod_{i=1}^{n} \left(\frac{1}{t^{\mu_i}} \frac{d}{dx}\right) y = y$$

i.e.

$$\prod_{i=1}^{n} \left(t^{-(1+\mu_i)} \theta\right) y = y$$

By repeated application of the shifting formula of Lemma 2 one can rewrite this equation as

$$\text{(A-11)} \qquad \prod_{k=0}^{n-1} (\theta - \sigma_k) y = t^{\sigma_n} y$$

where $\sigma_k = \sum_{i=1}^{k} (1 + \mu_i)$, $\sigma_0 = 1$.

Clearly this is a (general) hypergeometric equation. Moreover, from the criterion discussed above, (A-10) will have n independent regular solutions at t = 0 provided the μ_i are distinct non-negative integers.

When n = 2 (A-10) or, more generally the system

$$\text{(A-12)} \qquad \begin{aligned} \frac{d}{dt} g_1 &= t^{a_1} g_2 \\ \frac{d}{dt} g_2 &= t^{a_2} g_1 \end{aligned}$$

can be solved explicitly in terms of Bessel functions $J_{\pm\nu}(t)$ where [Watson, p.83]

$$(\theta + \nu)J_\nu(t) = tJ_{\nu-1}(t)$$
$$(\theta - \nu)J_\nu(t) = -tJ_{\nu+1}(t) \tag{A-13}$$

and, consequently,

$$(\theta^2 - \nu^2)J_{\pm\nu}(t) = -t^2 J_{\pm\nu}(t) \tag{A-14}$$

with the aid of Lemma 2, (A-13) can be rewritten as

$$\theta t^\nu J_{\pm\nu}(t) = \pm t^{1\pm\nu} J_{\pm(\nu-1)}(t) \tag{A-15}$$

The equation for g_1, corresponding to (A-11), is

$$\theta(\theta - (\alpha_1 + 1))g_1 = t^p g_1 \tag{A-16}$$

where

$$p = \alpha_1 + \alpha_2 + 2$$

Let

$$g_1 = t^{\frac{\alpha_1+1}{2}} h_1 \tag{A-17}$$

and

$$z = t^{\frac{p}{2}} \tag{A-18}$$

then, from (A-16) and Lemmas 2 and 3

$$\left[\theta_z^2 - \left(\frac{\alpha_1+1}{p}\right)^2\right]h_1 = -\left(\frac{2iz}{p}\right)^2 h_1$$

so that by (A-14)

$$h_1 = J_{\pm\left(\frac{\alpha_1+1}{p}\right)}\left(\frac{2i}{p} z\right) \tag{A-19}$$

In other words,

$$g_1 = z^{\frac{a_1+1}{p}} J_{\pm(\frac{a_1+1}{p})}(\tfrac{2i}{p} z) \tag{A-20}$$

For the determination of g_2, write the first equation of (A-12) as

$$g_2 = \tfrac{p}{2} z^{-\frac{2}{p}(a_1+1)} \theta_z g_1 \tag{A-21}$$

and apply (A-15) to obtain

$$g_2 = \pm \tfrac{p}{2} z^{\frac{a_1+1}{2}} J_{\mp(\frac{a_2+1}{p})}(\tfrac{2i}{p} z) \ . \tag{A-22}$$

Appendix B: Stokes Multipliers

Let U denote a matrix each of whose columns satisfies (1.1), i.e.,

$$\frac{d}{dx} U = [\rho A(x,\rho) + B(x,\rho)]U \tag{B-1}$$

Let $E(x,\rho)$ be defined as in (1.15), i.e.,

$$E(x,\rho) = \exp \rho \int_0^x J(t)dt \tag{B-2}$$

where $J(x)$ is a diagonal matrix which for x in $\mathcal{R}$ but $x \neq 0$ is the Jordan form of $A(x,\infty)$. Then (B-1) has a formal solution, for $x \neq 0$, of the form

$$U = \left[\sum_{j=0}^{\infty} \rho^{-j} V_j(x)\right] E(x,\rho) \tag{B-3}$$

where the columns of $V_o(x)$ are independent. Let $\tilde{\mathcal{R}}$ denote an annulus of $\mathcal{R}$ boundedly distant from the turning point $x = 0$. Then $\tilde{\mathcal{R}}$ can be covered by overlapping sectors $\tilde{\mathcal{S}}_i$ in each of which there exists an actual solution of (B-1) which is asymptotic to the formal solution uniformly in $\tilde{\mathcal{S}}_i$. Denote this actual solution by $\overset{i}{U}_\infty$ or, where convenient, by U_∞ with the superscript i understood. Let U_o denote an actual solution of (B-1) which reduces to the identity matrix at $x = 0$. From the linearity of (B-1) and the linear independence of the columns of U_∞ and U_o it follows that any solution U of (B-1) can be represented as

$$U = U_o M_o(U) = U_\infty M_\infty(U) \tag{B-4}$$

where the matrices $M_o(U)$ and $M_\infty(U)$ are independent of x. The elements of $M_\infty(U)$ are called Stokes multipliers of U with respect to U_∞ while the elements of $M_o(U)$ are called inverse Stokes multipliers of U with respect to U_o (Turrittin [1]).

One first evaluates inverse Stokes multipliers of U_∞ as follows. Apply (B-4) to U_∞ and evaluate each term at $x = 0$ to obtain

$$M_o(U_\infty) = U_\infty \quad \text{at } x = 0\ . \tag{B-5}$$

Thus the columns of $M_o(U_\infty)$ are connection formula and a typical column is evaluated in the manner indicated by the theorem in Chapter II.

Incidentally, Stokes multipliers for U_o can be determined from inverse Stokes for U_∞ as follows. Apply (B-4) to U_o and

evaluate each term at $x = 0$ to obtain, with the aid of (B-5),

$$M_\infty(U_o) = [M_o(U_\infty)]^{-1} \tag{B-6}$$

Finally, one notes from (B-4) and (B-5)

$$M_o(\overset{i}{U}_\infty)\overset{i}{M}_\infty(U) = M_o(\overset{j}{U}_\infty)\overset{j}{M}_\infty(U) \tag{B-7}$$

Thus, from a knowledge of Stokes multipliers for $\tilde{\delta}_i$ one obtains Stokes multipliers for $\tilde{\delta}_j$ as

$$\overset{j}{M}_\infty(U) = K\,\overset{i}{M}_\infty(U) \tag{B-8}$$

where

$$K = [M_o(\overset{j}{U}_\infty)]^{-1}\, M_o(\overset{i}{U}_\infty)$$

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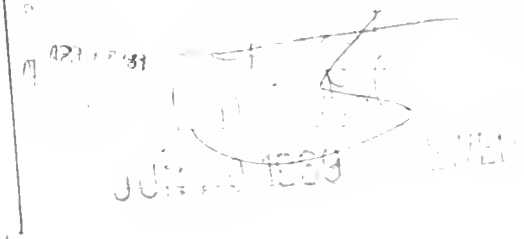

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